World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

Almost periodic solution of a discrete Lotka-Volterra mutualism model with time delays and feedback controls

Hui Zhang*

Mathematics and OR Section, Xi'an Research Institute of High-tech Hongqing Town, Xi'an, Shaanxi 710025, P R China

Abstract

In this paper, we consider an almost periodic discrete Lotka-Volterra mutualism model with delays. We first obtain the permanence and global attractivity of the system. By means of an almost periodic functional hull theory and constructing a suitable Lyapunov function, sufficient conditions are obtained for the existence of a unique strictly positive almost periodic solution which is globally attractive. An example together with numerical simulation indicates the feasibility of the main result.

Keywords: Almost periodic solution; mutualism model; Discrete; Delay; Permanence; Global attractivity

1 Introduction

In reference [1], Li had studied a discrete mutualism model with time delays:

$$\begin{cases} x_1(k+1) = x_1(k)\exp\left\{ r_1(k)\left[\dfrac{K_1(k)+\alpha_1(k)x_2(k-\tau_2(k))}{1+x_2(k-\tau_2(k))} - x_1(k-\sigma_1(k))\right]\right\}, \\ x_2(k+1) = x_2(k)\exp\left\{ r_2(k)\left[\dfrac{K_2(k)+\alpha_2(k)x_1(k-\tau_1(k))}{1+x_1(k-\tau_1(k))} - x_2(k-\sigma_2(k))\right]\right\}, \end{cases} \tag{1.1}$$

where $\{r_i(k)\}, \{K_i(k)\}, \{\alpha_i(k)\}, \{\tau_i(k)\}$ and $\{\sigma_i(k)\}$, with $i = 1, 2$, are positive ω-periodic sequence, and $\alpha_i > K_i$. By using the theory of coincidence degree theory, it is proved that system (1.1) has at least one positive ω-periodic solution.

Under the assumptions that $r_i, K_i, \alpha_i, \tau_i$ and σ_i, with $i = 1, 2$, are non-negative sequence bounded above and below by positive constants, and $\alpha_i > K_i,\ i = 1, 2$, Chen [2] obtained sufficient conditions that ensure the permanence of the system (1.1). To the best of the author's knowledge, though many works have been done for the mutualism model with time delays [3–5], most of the works dealt with the continuous time model. For more results about the existence of almost periodic solutions of a continuous time system, we can refer to [6–9] and the references cited therein. To this day, still no scholars have considered discrete almost periodic mutualism system with delays.

In this paper, we study the following discrete Lotka-Volterra mutualism model with delays and feedback control

$$\begin{cases} x_1(n+1) = x_1(n)\exp\left\{ a_1(n) - b_1(n)x_1(n-\sigma_1) + c_1(n)\dfrac{x_2(n-\tau_1)}{d_1(n)+x_2(n-\tau_1)} - e_1(n)u_1(n-\delta_1)\right\}, \\ x_2(n+1) = x_2(n)\exp\left\{ a_2(n) - b_2(n)x_2(n-\sigma_2) + c_2(n)\dfrac{x_1(n-\tau_2)}{d_2(n)+x_1(n-\tau_2)} - e_2(n)u_2(n-\delta_2)\right\}, \\ \Delta u_1(n) = -f_1(n) + g_1(n)x_1(n-\eta_1), \\ \Delta u_2(n) = -f_2(n) + g_2(n)x_2(n-\eta_2) \end{cases} \tag{1.2}$$

*Corresponding author. E-mail address: zh53054958@163.com

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

where $\{a_i(n)\}, \{b_i(n)\}$ and $\{c_i(n)\}(i = 1, 2)$ are bounded nonnegative almost periodic sequences such that

$$0 < a_i^l \le a_i(n) \le a_i^u, \qquad 0 < b_i^l \le b_i(n) \le b_i^u, \qquad 0 < c_i^l \le c_i(n) \le c_i^u,$$

where $i = 1, 2,\ n \in Z$. For any bounded sequence $f(n)$ defined on Z, $f^u = \sup\limits_{n\in Z} f(n), f^l = \inf\limits_{n\in Z} f(n)$. Also τ_i and $\sigma_i(i = 1, 2)$ are positive integers.

By the biological meaning, we will focus our discussion on the positive solutions of system (1.2). So it is assumed that the initial conditions of system (1.2) are the form:

$$x_i(\theta) = \varphi_i(\theta) \ge 0, \quad \varphi_i(0) > 0, \quad \theta \in N[-\tau, 0] = \{-\tau, -\tau + 1, \ldots, 0\}, \quad \tau = \max\{\tau_1, \sigma_1, \tau_2, \sigma_2\}. \tag{1.3}$$

With the stimulation from the works [10–18], the main purpose of this paper is to obtain a set of sufficient conditions to ensure the existence of a unique globally attractive positive almost periodic solution of system (1.2) with initial condition (1.3).

The remaining part of this paper is organized as follows: In Section 2, we will introduce some definitions and several useful lemmas. In the next section, we establish the permanence of system (1.2). Sufficient conditions for the global attractivity of system (1.2) are showed in Section 4. Then, in Section 5, we establish sufficient conditions to ensure the existence of a unique strictly positive almost periodic solution, which is globally attractive. The main result is illustrated by an example with a numerical simulation in the last section.

2 Preliminaries

First, we give the definitions of the terminologies involved.

Definition 2.1([19]) A sequence $x : Z \to R$ is called an almost periodic sequence if the ε-translation set of x

$$E\{\varepsilon, x\} = \{\tau \in Z :\mid x(n + \tau) - x(n) \mid< \varepsilon, \forall n \in Z\}$$

is a relatively dense set in Z for all $l(\varepsilon) > 0$; that is, for any given $l(\varepsilon) > 0$, there exists an integer $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains an integer $\tau \in E\{\varepsilon, x\}$ with

$$\mid x(n + \tau) - x(n) \mid< \varepsilon, \quad \forall n \in Z.$$

τ is called an ε-translation number of $x(n)$.

Definition 2.2([20]) Let D be an open subset of R^m, $f : Z \times D \to R^m$. $f(n, x)$ is said to be almost periodic in n uniformly for $x \in D$ if for any $\varepsilon > 0$ and any compact set $S \subset D$, there exists a positive integer $l = l(\varepsilon, S)$ such that any interval of length l contains an integer τ for which

$$|f(n + \tau, x) - f(n, x)| < \varepsilon, \quad \forall (n, x) \in Z \times S.$$

τ is called an ε-translation number of $f(n, x)$.

Definition 2.3([21]) The hull of f, denoted by $H(f)$, is defined by

$$H(f) = \{g(n, x) : \lim_{k\to\infty} f(n + \tau_k, x) = g(n, x) \text{ uniformly on } Z \times S\},$$

for some sequence $\{\tau_k\}$, where S is any compact set in D.

Definition 2.4 Suppose that $X(n) = (x_1(n), x_2(n))$ is any solution of system (1.1). $X(n)$ is said to be a strictly positive solution in Z if for $n \in Z$ and $i = 1, 2$

$$0 < \inf_{n\in Z} x_i(n) \le \sup_{n\in Z} x_i(n) < \infty.$$

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

Now, we state several lemmas which will be useful in proving our main result.

Lemma 2.1([22]) $\{x(n)\}$ is an almost periodic sequence if and only if for any integer sequence $\{k_i'\}$, there exists a subsequence $\{k_i\} \subset \{k_i'\}$ such that the sequence $\{x(n+k_i)\}$ converges uniformly for all $n \in Z$ as $i \to \infty$. Furthermore, the limit sequence is also an almost periodic sequence.

Lemma 2.2([2]) Assume that $\{x(n)\}$ satisfies $x(n) > 0$ and

$$x(n+1) \le x(n)\exp\{a(n) - b(n)x(n)\}$$

for $n \in N$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants. Then

$$\limsup_{n\to+\infty} x(n) \le \frac{1}{b^l}\exp(a^u - 1).$$

Lemma 2.3([2]) Assume that $\{x(n)\}$ satisfies

$$x(n+1) \ge x(n)\exp\{a(n) - b(n)x(n)\}, \quad n \ge N_0,$$

$$\limsup_{n\to+\infty} x(n) \le x^*$$

and $x(N_0) > 0$, where $a(n)$ and $b(n)$ are non-negative sequences bounded above and below by positive constants and $N_0 \in N$. Then

$$\liminf_{n\to+\infty} x(n) \ge \min\left\{\frac{a^l}{b^u}\exp\{a^l - b^u x^*\}, \frac{a^l}{b^u}\right\}.$$

3 Permanence

In this section, we establish the permanence result for system (1.2).

Theorem 3.1 System (1.2) with initial condition (1.3) is permanent, that is, there exist positive constants m_i and $M_i (i = 1, 2)$ which are independent of the solutions of system (1.2), such that for any positive solution $(x_1(n), x_2(n))$ of system (1.2), one has:

$$m_i \le \liminf_{n\to+\infty} x_i(n) \le \limsup_{n\to+\infty} x_i(n) \le M_i, \qquad i = 1, 2.$$

Proof. Let $(x_1(n), x_2(n))$ be any positive solution of system (1.2) with initial condition (1.3). From the first equation of system (1.2) it follows that

$$x_1(n+1) \le x_1(n)\exp\big\{a_1(n) + c_1(n)\big\} \le x_1(n)\exp\big\{a_1^u + c_1^u\big\}. \tag{3.1}$$

By using (3.1), one could easily obtain that

$$x_1(n-\sigma_1) \ge x_1(n)\exp\big\{-\sigma_1(a_1^u + c_1^u)\big\}. \tag{3.2}$$

Substituting (3.2) into the first equation of system (1.2), it follows that

$$x_1(n+1) \le x_1(n)\exp\bigg\{a_1^u + c_1^u - b_1^l\exp\big\{-\sigma_1(a_1^u + c_1^u)\big\}x_1(n)\bigg\}. \tag{3.3}$$

Thus, as a direct corollary of Lemma 2.2, according to (3.3), one has

$$\limsup_{n\to+\infty} x_1(n) \le \frac{1}{b_1^l}\exp\{(a_1^u + c_1^u)(\sigma_1 + 1) - 1\} \triangleq M_1. \tag{3.4}$$

By using the second equation of system (1.2), similar to the analysis of (3.1)-(3.4), we can obatin

$$\limsup_{n\to+\infty} x_2(n) \le \frac{1}{b_2^l}\exp\{(a_2^u + c_2^u)(\sigma_2 + 1) - 1\} \triangleq M_2. \tag{3.5}$$

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

For any small positive constant $\varepsilon > 0$, from (3.4) and (3.5) it follows that there exists a $N_1 > 0$ such that for all $n > N_1$ and $i = 1, 2$,

$$x_i(n) \leq M_i + \varepsilon. \tag{3.6}$$

For $n \geq N_1 + \sigma_1$, from (3.6) and the first equation of system (1.2), we have

$$x_1(n+1) \geq x_1(n) \exp\big\{a_1(n) - b_1(n)x_1(n-\sigma_1)\big\} \geq x_1(n) \exp\big\{a_1^l - b_1^u(M_1+\varepsilon)\big\}. \tag{3.7}$$

Thus, by using (3.7) we obtain

$$x_1(n-\sigma_1) \leq x_1(n) \exp\big\{-\sigma_1[a_1^l - b_1^u(M_1+\varepsilon)]\big\}. \tag{3.8}$$

Substituting (3.8) into the first equation of system (1.2), for $n \geq N_1 + \sigma_1$, it follows that

$$x_1(n+1) \geq x_1(n) \exp\Big\{a_1^l - b_1^u \exp\big\{-\sigma_1[a_1^l - b_1^u(M_1+\varepsilon)]\big\}x_1(n)\Big\}. \tag{3.9}$$

Thus, as a direct corollary of Lemma 2.3, according to (3.4) and (3.9), one has

$$\liminf_{n\to+\infty} x_1(n) \geq \min\{A_{1\varepsilon}, A_{2\varepsilon}\}, \tag{3.10}$$

where

$$\begin{aligned} A_{1\varepsilon} &= \frac{a_1^l}{b_1^u} \exp\big\{\sigma_1[a_1^l - b_1^u(M_1+\varepsilon)]\big\}, \\ A_{2\varepsilon} &= A_{1\varepsilon} \exp\Big\{a_1^l - b_1^u \exp\big\{-\sigma_1[a_1^l - b_1^u(M_1+\varepsilon)]\big\}M_1\Big\}. \end{aligned}$$

Letting $\varepsilon \to 0$, it follows that

$$\liminf_{n\to+\infty} x_1(n) \geq \frac{1}{2}\min\{A_1, A_2\} \triangleq m_1 > 0, \tag{3.11}$$

where

$$\begin{aligned} A_1 &= \frac{a_1^l}{b_1^u} \exp\big\{\sigma_1(a_1^l - b_1^u M_1)\big\}, \\ A_2 &= A_1 \exp\Big\{a_1^l - b_1^u \exp\big\{-\sigma_1(a_1^l - b_1^u M_1)\big\}M_1\Big\}. \end{aligned}$$

Similar to the analysis of (3.7)-(3.11), by applying (3.6), from the second equation of system (1.2), we also have that

$$\liminf_{n\to+\infty} x_2(n) \geq \frac{1}{2}\min\{B_1, B_2\} \triangleq m_2 > 0, \tag{3.12}$$

where

$$\begin{aligned} B_1 &= \frac{a_2^l}{b_2^u} \exp\big\{\sigma_2(a_2^l - b_2^u M_2)\big\}, \\ B_2 &= B_1 \exp\Big\{a_2^l - b_2^u \exp\big\{-\sigma_2(a_2^l - b_2^u M_2)\big\}M_2\Big\}. \end{aligned}$$

Then, (3.4),(3.5) and (3.11),(3.12) show that system (1.2) is permanent. The proof is completed.

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

4 Global attractivity

In this section, by constructing a non-negative Lyapunov-like functional, we will obtain sufficient conditions for global attractivity of positive solutions of system (1.2) with initial condition (1.3). We first introduce a definition and prove a lemma which will be useful to obtain our main result.

Definition 4.1 A solution $(x_1(n), x_2(n))$ of system (1.2) with initial condition (1.3) is said to be globally attractive if for any other solution $(x_1^*(n), x_2^*(n))$ of system (1.2) with initial condition (1.3), we have

$$\lim_{n\to+\infty}(x_i^*(n) - x_i(n)) = 0, i = 1, 2.$$

Lemma 4.1 For any two positive solutions $(x_1(n), x_2(n))$ and $(x_1^*(n), x_2^*(n))$ of system (1.2) with initial condition (1.3), we have

$$\begin{aligned}\ln\frac{x_i(n+1)}{x_i^*(n+1)} &= \ln\frac{x_i(n)}{x_i^*(n)} - b_i(n)[x_i(n) - x_i^*(n)] + c_i(n)\left[\frac{1}{1+x_j^*(n-\tau_i)} - \frac{1}{1+x_j(n-\tau_i)}\right]\\ &+b_i(n)\sum_{s=n-\sigma_i}^{n-1}\Big\{[x_i(s) - x_i^*(s)]A_i(s)[a_i(s) - b_i(s)x_i^*(s-\sigma_i) + c_i(s)\frac{x_j^*(s-\tau_i)}{1+x_j^*(s-\tau_i)}]\\ &+ x_i(s)B_i(s)\left[c_i(s)[\frac{1}{1+x_j^*(s-\tau_i)} - \frac{1}{1+x_j(s-\tau_i)}] - b_i(s)[x_i(s-\sigma_i) - x_i^*(s-\sigma_i)]\right]\Big\}, \quad (4.1)\end{aligned}$$

where

$$\begin{aligned}A_i(s) &= \exp\big\{\theta_i(s)[a_i(s) - b_i(s)x_i^*(s-\sigma_i) + c_i(s)\frac{x_j^*(s-\tau_i)}{1+x_j^*(s-\tau_i)}]\big\},\\ B_i(s) &= \exp\Big\{\varphi_i(s)[a_i(s) - b_i(s)x_i(s-\sigma_i) + c_i(s)\frac{x_j(s-\tau_i)}{1+x_j(s-\tau_i)}]\\ &\quad + (1-\varphi_i(s))\big[a_i(s) - b_i(s)x_i^*(s-\sigma_i) + c_i(s)\frac{x_j^*(s-\tau_i)}{1+x_j^*(s-\tau_i)}]\Big\}, \quad (4.2)\end{aligned}$$

$\theta_i(s), \varphi_i(s) \in (0, 1), i \neq j; i, j = 1, 2.$

Proof. For $i \neq j; i, j = 1, 2$, we can have from(1.2)

$$\begin{aligned}&\ln\frac{x_i(n+1)}{x_i^*(n+1)} - \ln\frac{x_i(n)}{x_i^*(n)} = \ln\frac{x_i(n+1)}{x_i(n)} - \ln\frac{x_i^*(n+1)}{x_i^*(n)}\\ &= a_i(n) - b_i(n)x_i(n-\sigma_i) + c_i(n)\frac{x_j(n-\tau_i)}{1+x_j(n-\tau_i)} - \left[a_i(n) - b_i(n)x_i^*(n-\sigma_i) + c_i(n)\frac{x_j^*(n-\tau_i)}{1+x_j^*(n-\tau_i)}\right]\\ &= c_i(n)\left[\frac{x_j(n-\tau_i)}{1+x_j(n-\tau_i)} - \frac{x_j^*(n-\tau_i)}{1+x_j^*(n-\tau_i)}\right] - b_i(n)[x_i(n-\sigma_i) - x_i^*(n-\sigma_i)]\\ &= c_i(n)\left[\frac{1}{1+x_j^*(n-\tau_i)} - \frac{1}{1+x_j(n-\tau_i)}\right] - b_i(n)[x_i(n) - x_i^*(n)]\\ &\quad + b_i(n)\{[x_i(n) - x_i(n-\sigma_i)] - [x_i^*(n) - x_i^*(n-\sigma_i)]\},\end{aligned}$$

that is

$$\begin{aligned}\ln\frac{x_i(n+1)}{x_i(n)} &= \ln\frac{x_i^*(n+1)}{x_i^*(n)} + c_i(n)\left[\frac{1}{1+x_j^*(n-\tau_i)} - \frac{1}{1+x_j(n-\tau_i)}\right]\\ &-b_i(n)[x_i(n) - x_i^*(n)] + b_i(n)\big\{[x_i(n) - x_i(n-\sigma_i)] - [x_i^*(n) - x_i^*(n-\sigma_i)]\big\}. \quad (4.3)\end{aligned}$$

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

Since

$$\begin{aligned}
&[x_i(n)-x_i(n-\sigma_i)]-[x_i^*(n)-x_i^*(n-\sigma_i)]\\
&=\sum_{s=n-\sigma_i}^{n-1}[x_i(s+1)-x_i(s)]-\sum_{s=n-\sigma_i}^{n-1}[x_i^*(s+1)-x_i^*(s)]\\
&=\sum_{s=n-\sigma_i}^{n-1}\{[x_i(s+1)-x_i^*(s+1)]-[x_i(s)-x_i^*(s)]\},
\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}
&[x_i(s+1)-x_i^*(s+1)]-[x_i(s)-x_i^*(s)]\\
&=x_i(s)\exp\left\{a_i(s)-b_i(s)x_i(s-\sigma_i)+c_i(s)\frac{x_j(s-\tau_i)}{1+x_j(s-\tau_i)}\right\}\\
&\quad-x_i^*(s)\exp\left\{a_i(s)-b_i(s)x_i^*(s-\sigma_i)+c_i(s)\frac{x_j^*(s-\tau_i)}{1+x_j^*(s-\tau_i)}\right\}-[x_i(s)-x_i^*(s)]\\
&=[x_i(s)-x_i^*(s)]\left\{\exp\left[a_i(s)-b_i(s)x_i^*(s-\sigma_i)+c_i(s)\frac{x_j^*(s-\tau_i)}{1+x_j^*(s-\tau_i)}\right]-1\right\}\\
&\quad+x_i(s)\left\{\exp\left[a_i(s)-b_i(s)x_i(s-\sigma_i)+c_i(s)\frac{x_j(s-\tau_i)}{1+x_j(s-\tau_i)}\right]\right.\\
&\qquad\left.-\exp\left[a_i(s)-b_i(s)x_i^*(s-\sigma_i)+c_i(s)\frac{x_j^*(s-\tau_i)}{1+x_j^*(s-\tau_i)}\right]\right\}.
\end{aligned}$$

Using the Mean Value Theorem, we get

$$\begin{aligned}
&[x_i(s+1)-x_i^*(s+1)]-[x_i(s)-x_i^*(s)]\\
&=[x_i(s)-x_i^*(s)]A_i(s)\left[a_i(s)-b_i(s)x_i^*(s-\sigma_i)+c_i(s)\frac{x_j^*(s-\tau_i)}{1+x_j^*(s-\tau_i)}\right]\\
&\quad+x_i(s)B_i(s)\left[c_i(s)\left[\frac{1}{1+x_j^*(s-\tau_i)}-\frac{1}{1+x_j(s-\tau_i)}\right]-b_i(s)[x_i(s-\sigma_i)-x_i^*(s-\sigma_i)]\right],
\end{aligned}\tag{4.5}$$

here $A_i(s), B_i(s)$ are defined by (4.2). Then from (4.3)-(4.5), we can easily obtain (4.1). The proof is completed.

Theorem 4.1 Assume that in system (1.2) with initial condition (1.3), there exist positive constants β_1, β_2 and $\eta>0$ such that

$$\beta_i E_{ij}-\beta_j F_j\ge\eta,\quad i,j=1,2,j\neq i\tag{4.6}$$

where

$$\begin{aligned}
&E_{ij}=\min\{b_i^l,\frac{2}{M_i}-b_i^u\}-\sigma_iM_i(b_i^u)^2B_i^u-\sigma_ib_i^uA_i^u(a_i^u+b_i^uM_i+c_i^uM_j);\\
&F_j=c_j^u+\sigma_jM_jb_j^uB_j^uc_j^u.
\end{aligned}\tag{4.7}$$

Then for any two positive solutions $(x_1(n),x_2(n))$ and $(x_1^*(n),x_2^*(n))$ of system (1.2) with initial condition (1.3), we have

$$\lim_{n\to+\infty}(x_i^*(n)-x_i(n))=0,\quad i=1,2.$$

Proof. Firstly, let $V_{11}(n)=\big|\ln x_1(n)-\ln x_1^*(n)\big|$. From (4.1), we have that

$$\begin{aligned}
&\Big|\ln\frac{x_1(n+1)}{x_1^*(n+1)}\Big|\le\Big|\ln\frac{x_1(n)}{x_1^*(n)}-b_1(n)[x_1(n)-x_1^*(n)]\Big|+c_1(n)\big|x_2(n-\tau_1)-x_2^*(n-\tau_1)\big|\\
&\quad+b_1(n)\sum_{s=n-\sigma_1}^{n-1}\Big\{\big|x_1(s)-x_1^*(s)\big|A_1(s)[a_1(s)+b_1(s)\big|x_1^*(s-\sigma_1)\big|+c_1(s)\big|x_2^*(s-\tau_1)\big|]\\
&\quad+\big|x_1(s)\big|B_1(s)\big[c_1(s)\big|x_2(s-\tau_1)-x_2^*(s-\tau_1)\big|+b_1(s)\big|x_1(s-\sigma_1)-x_1^*(s-\sigma_1)\big|\big]\Big\},
\end{aligned}\tag{4.8}$$

Since

$$x_i(n) - x_i^*(n) = e^{\ln x_i(n)} - e^{\ln x_i^*(n)} = \xi_i(n)\ln(x_i(n)/x_i^*(n)), \quad i = 1, 2,$$

where $\xi_i(n)$ lies between $x_i(n)$ and $x_i^*(n), i = 1, 2$, it follows that

$$\begin{aligned} &\big|\ln(x_1(n)/x_1^*(n)) - b_1(n)[x_1(n) - x_1^*(n)]\big| \\ &= \big|\ln(x_1(n)/x_1^*(n)) - b_1(n)\xi_1(n)\ln(x_1(n)/x_1^*(n))\big| \\ &= \big|\ln(x_1(n)/x_1^*(n))\big| - \Big(\frac{1}{\xi_1(n)} - \Big|\frac{1}{\xi_1(n)} - b_1(n)\Big|\Big)\big|x_1(n) - x_1^*(n)\big|. \end{aligned} \tag{4.9}$$

By Theorem 3.1, there are constants $M_i > 0 (i = 1, 2)$, and a positive integer n_0 such that for $n > n_0, 0 < x_i(n), x_i^*(n) \le M_i (i = 1, 2)$. Then from (4.7) and (4.8) we can obtain that for $n \ge n_0 + \tau$,

$$\begin{aligned} \triangle V_{11} \le & -\Big(\frac{1}{\xi_1(n)} - \Big|\frac{1}{\xi_1(n)} - b_1(n)\Big|\Big)\big|x_1(n) - x_1^*(n)\big| + c_1(n)\big|x_2(n-\tau_1) - x_2^*(n-\tau_1)\big| \\ & + b_1(n)\sum_{s=n-\sigma_1}^{n-1}\Big\{A_1(s)[a_1(s) + M_1 b_1(s) + M_2 c_1(s)]\big|x_1(s) - x_1^*(s)\big| \\ & + M_1 B_1(s) c_1(s)\big|x_2(s-\tau_1) - x_2^*(s-\tau_1)\big| + M_1 B_1(s) b_1(s)\big|x_1(s-\sigma_1) - x_1^*(s-\sigma_1)\big|\Big\}. \end{aligned} \tag{4.10}$$

Secondly, let

$$\begin{aligned} V_{12}(n) = & \sum_{s=n-\tau_1}^{n-1} c_1(s+\tau_1)\big|x_2(s) - x_2^*(s)\big| \\ & + \sum_{s=n}^{n-1+\sigma_1} b_1(s)\sum_{u=s-\sigma_1}^{n-1}\Big\{A_1(u)[a_1(u) + M_1 b_1(u) + M_2 c_1(u)]\big|x_1(u) - x_1^*(u)\big| \\ & + M_1 B_1(u) c_1(u)\big|x_2(u-\tau_1) - x_2^*(u-\tau_1)\big| \\ & + M_1 B_1(u) b_1(u)\big|x_1(u-\sigma_1) - x_1^*(u-\sigma_1)\big|\Big\}. \end{aligned} \tag{4.11}$$

By a simple calculation, we can obtain

$$\begin{aligned} \triangle V_{12} = & c_1(n+\tau_1)\big|x_2(n) - x_2^*(n)\big| - c_1(n)\big|x_2(n-\tau_1) - x_2^*(n-\tau_1)\big| \\ & + \sum_{s=n+1}^{n+\sigma_1} b_1(s)\Big\{A_1(n)[a_1(n) + M_1 b_1(n) + M_2 c_1(n)]\big|x_1(n) - x_1^*(n)\big| \\ & + M_1 B_1(n) c_1(n)\big|x_2(n-\tau_1) - x_2^*(n-\tau_1)\big| \\ & + M_1 B_1(n) b_1(n)\big|x_1(n-\sigma_1) - x_1^*(n-\sigma_1)\big|\Big\} \\ & - b_1(n)\sum_{u=n-\sigma_1}^{n-1}\Big\{A_1(u)[a_1(u) + M_1 b_1(u) + M_2 c_1(u)]\big|x_1(u) - x_1^*(u)\big| \\ & + M_1 B_1(u) c_1(u)\big|x_2(u-\tau_1) - x_2^*(u-\tau_1)\big| \\ & + M_1 B_1(u) b_1(u)\big|x_1(u-\sigma_1) - x_1^*(u-\sigma_1)\big|\Big\}. \end{aligned} \tag{4.12}$$

Thirdly,let

$$\begin{aligned} V_{13}(n) = & M_1\sum_{l=n-\tau_1}^{n-1} B_1(l+\tau_1)c_1(l+\tau_1)\big|x_2(l) - x_2^*(l)\big|\sum_{s=l+\tau_1+1}^{l+\tau_1+\sigma_1} b_1(s) \\ & + M_1\sum_{l=n-\sigma_1}^{n-1} B_1(l+\sigma_1)b_1(l+\sigma_1)\big|x_1(l) - x_1^*(l)\big|\sum_{s=l+\sigma_1+1}^{l+2\sigma_1} b_1(s). \end{aligned}$$

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

Then we can derive

$$\begin{aligned}\triangle V_{13} = {} & M_1 \sum_{s=n+\tau_1+1}^{n+\tau_1+\sigma_1} b_1(s)B_1(n+\tau_1)c_1(n+\tau_1)\big|x_2(n)-x_2^*(n)\big| \\ & - M_1 \sum_{s=n+1}^{n+\sigma_1} b_1(s)B_1(n)c_1(n)\big|x_2(n-\tau_1)-x_2^*(n-\tau_1)\big| \\ & + M_1 \sum_{s=n+\sigma_1+1}^{n+2\sigma_1} b_1(s)B_1(n+\sigma_1)b_1(n+\sigma_1)\big|x_1(n)-x_1^*(n)\big| \\ & - M_1 \sum_{s=n+1}^{n+\sigma_1} b_1(s)B_1(n)b_1(n)\big|x_1(n-\sigma_1)-x_1^*(n-\sigma_1)\big| \qquad (4.13)\end{aligned}$$

Now we set $V_1(n) = V_{11}(n) + V_{12}(n) + V_{13}(n)$. Then from (4.8)-(4.13), we have that for $n \geq n_0 + \tau$,

$$\begin{aligned}\triangle V_1 \leq {} & -\Big(\frac{1}{\xi_1(n)} - \Big|\frac{1}{\xi_1(n)} - b_1(n)\Big|\Big)\big|x_1(n)-x_1^*(n)\big| + c_1(n+\tau_2)\big|x_2(n)-x_2^*(n)\big| \\ & + \sum_{s=n+1}^{n+\sigma_1} b_1(s)A_1(n)[a_1(n)+M_1b_1(n)+M_2c_1(n)]\big|x_1(n)-x_1^*(n)\big| \\ & + \sum_{s=n+\tau_1+1}^{n+\tau_1+\sigma_1} b_1(s)M_1B_1(n+\tau_1)c_1(n+\tau_1)\big|x_2(n)-x_2^*(n)\big| \\ & + \sum_{s=n+\sigma_1+1}^{n+2\sigma_1} b_1(s)M_1B_1(n+\sigma_1)b_1(n+\sigma_1)\big|x_1(n)-x_1^*(n)\big|.\end{aligned}$$

By arguments similar to those above, we take

$$\begin{aligned}V_{21}(n) = {} & \big|\ln x_2(n) - \ln x_2^*(n)\big|, \\ V_{22}(n) = {} & \sum_{s=n-\tau_2}^{n-1} c_2(s+\tau_2)\big|x_1(s)-x_1^*(s)\big| \\ & + \sum_{s=n}^{n-1+\sigma_2} b_1(s) \sum_{u=s-\sigma_2}^{n-1} \Big\{A_2(u)[a_2(u)+M_2b_2(u)+M_1c_2(u)]\big|x_2(u)-x_2^*(u)\big| \\ & + M_2B_2(u)c_2(u)\big|x_1(u-\tau_2)-x_1^*(u-\tau_2)\big| + M_2B_2(u)b_2(u)\big|x_2(u-\sigma_2)-x_2^*(u-\sigma_2)\big|\Big\}, \\ V_{23}(n) = {} & M_2 \sum_{l=n-\tau_2}^{n-1} B_2(l+\tau_2)c_2(l+\tau_2)\big|x_1(l)-x_1^*(l)\big| \sum_{s=l+\tau_2+1}^{l+\tau_2+\sigma_2} b_2(s) \\ & + M_2 \sum_{l=n-\sigma_2}^{n-1} B_2(l+\sigma_2)b_2(l+\sigma_2)\big|x_2(l)-x_2^*(l)\big| \sum_{s=l+\sigma_2+1}^{l+2\sigma_2} b_2(s).\end{aligned}$$

Similarly, we take $V_2(n) = V_{21}(n) + V_{22}(n) + V_{23}(n)$, then in the same way as obtaining $\triangle V_1$, we can obtain for $n \geq n_0 + \tau$,

$$\begin{aligned}\triangle V_2(n) \leq {} & -\Big(\frac{1}{\xi_2(n)} - \Big|\frac{1}{\xi_2(n)} - b_2(n)\Big|\Big)\big|x_2(n)-x_2^*(n)\big| + c_2(n+\tau_1)\big|x_1(n)-x_1^*(n)\big| \\ & + \sum_{s=n+1}^{n+\sigma_2} b_2(s)A_2(n)[a_2(n)+M_2b_2(n)+M_1c_2(n)]\big|x_2(n)-x_2^*(n)\big| \\ & + \sum_{s=n+\tau_2+1}^{n+\tau_2+\sigma_2} b_2(s)M_2B_2(n+\tau_2)c_2(n+\tau_2)\big|x_1(n)-x_1^*(n)\big| \\ & + \sum_{s=n+\sigma_2+1}^{n+2\sigma_2} b_2(s)M_2B_2(n+\sigma_2)b_2(n+\sigma_2)\big|x_2(n)-x_2^*(n)\big|.\end{aligned}$$

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

Now we define a Lyapunov-like discrete functional V by

$$V(n) = \beta_1 V_1(n) + \beta_2 V_2(n).$$

It is easy to see that $V(n_0+\tau) < +\infty$. Calculating the difference of V along the solution of (1.2) with initial condition (1.3), we have that for $n \geq n_0+\tau$,

$$\begin{aligned}
\triangle V(n) \leq & -\sum_{i=1}^{2}\Bigg\{\beta_i\Big[\Big(\frac{1}{\xi_i(n)} - \Big|\frac{1}{\xi_i(n)} - b_i(n)\Big|\Big) - \sum_{s=n+\sigma_i+1}^{n+2\sigma_i} b_i(s)M_iB_i(n+\sigma_i)b_i(n+\sigma_i)\Big] \\
& - \Big[\beta_i \sum_{s=n+1}^{n+\sigma_i} b_i(s)A_i(n)[a_i(n)+M_ib_i(n)+M_jc_i(n)] \\
& + \beta_j[c_j(n+\tau_j) + \sum_{s=n+\tau_j+1}^{n+\tau_j+\sigma_j} b_j(s)M_jB_j(n+\tau_j)c_j(n+\tau_j)]\Big]\Bigg\}\big|x_i(n)-x_i^*(n)\big| \\
\leq & -\sum_{i=1}^{2}\Bigg\{\beta_i\Big[\min\{b_i^l, \frac{2}{M_i} - b_i^u\} - \sigma_iM_i(b_i^u)^2B_i^u\Big] \\
& - \Big[\beta_i\sigma_ib_i^uA_i^u(a_i^u + b_i^uM_i + c_i^uM_j) + \beta_j(c_j^u + \sigma_jM_jb_j^uB_j^uc_j^u)\Big]\Bigg\}\big|x_i(n)-x_i^*(n)\big| \\
= & -\sum_{i=1}^{2}(\beta_iE_{ij} - \beta_jF_j)\big|x_i(n)-x_i^*(n)\big| \\
\leq & -\eta\sum_{i=1}^{2}\big|x_i(n)-x_i^*(n)\big|, \quad j=1,2, j\neq i,
\end{aligned}$$

where E_{ij} and F_j are defined by (4.7).
Then we have that

$$\sum_{p=n_0+\tau}^{n}[V(p+1)-V(p)] \leq -\eta\sum_{p=n_0+\tau}^{n}\sum_{i=1}^{2}\big|x_i(p)-x_i^*(p)\big|,$$

which implies

$$V(n+1) + \eta\sum_{p=n_0+\tau}^{n}\sum_{i=1}^{2}\big|x_i(p)-x_i^*(p)\big| \leq V(n_0+\tau).$$

That is

$$\sum_{p=n_0+\tau}^{n}\sum_{i=1}^{2}\big|x_i(p)-x_i^*(p)\big| \leq \frac{V(n_0+\tau)}{\eta}$$

and then

$$\sum_{n=n_0+\tau}^{\infty}\sum_{i=1}^{2}\big|x_i(n)-x_i^*(n)\big| \leq \frac{V(n_0+\tau)}{\eta} < +\infty$$

which means that $\lim\limits_{n\to+\infty}\sum\limits_{i=1}^{2}\big|x_i(n)-x_i^*(n)\big| = 0$, that is

$$\lim_{n\to+\infty}(x_i(n)-x_i^*(n)) = 0, \quad i=1,2.$$

It means that $(x_1(n), x_2(n))$ is globally attractive. This completes the proof of Theorem 4.1.

5 Almost periodic solution

In this section, we will study the existence of a globally attractive almost periodic sequence solution of system (1.2) with initial condition (1.3) by means of an almost periodic functional hull theory and constructing a suitable Lyapunov function, and obtain the sufficient conditions.

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

Let $\{\delta_k\}$ be any integer valued sequence such that $\delta_k \to \infty$ as $k \to \infty$. According to Lemma 2.1, taking a subsequence if necessary, we have $a_i(n+\delta_k) \to a_i^*(n), b_i(n+\delta_k) \to b_i^*(n), c_i(n+\delta_k) \to c_i^*(n),\ i=1,2$, as $k \to \infty$ for $n \in Z$. Then we get a hull equation of system (1.2) as follows:

$$\begin{cases} x_1(n+1) = x_1(n)\exp\left\{a_1^*(n) - b_1^*(n)x_1(n-\sigma_1) + c_1^*(n)\dfrac{x_2(n-\tau_1)}{1+x_2(n-\tau_1)}\right\}, \\ x_2(n+1) = x_2(n)\exp\left\{a_2^*(n) - b_2^*(n)x_2(n-\sigma_2) + c_2^*(n)\dfrac{x_1(n-\tau_2)}{1+x_1(n-\tau_2)}\right\}, \end{cases} \tag{5.1}$$

By the almost periodic theory, we can conclude that if system (1.2) satisfies (4.6), then the hull equation (5.1) of system (1.2) also satisfies (4.6).

By Theorem 3.4 in [18], we can easily obtain the lemma as follows.

Lemma 5.1 If each hull equation of system (1.2) has a unique strictly positive solution, then the almost periodic difference system (1.2) has a unique strictly positive almost periodic solution.

Theorem 5.1 If the almost periodic difference system (1.2) satisfies (4.6), then the almost periodic difference system (1.2) admits a unique strictly positive almost periodic solution, which is globally attractive.

Proof. By Lemma 5.1, we only need to prove that each hull equation of system (1.1) has a unique globally attractive almost periodic sequence solution; hence we firstly prove that each hull equation of system (1.1) has at least one strictly positive solution(the existence), and then we prove that each hull equation of system (1.1) has a unique strictly positive solution(the uniqueness).

Now we prove the existence of a strictly positive solution of any hull equation (5.1). By the almost periodicity of $\{a_i(n)\}, \{b_i(n)\}$ and $\{c_i(n)\}, i=1,2$, there exists an integer valued sequence $\{\tau_k\}$ with $\tau_k \to \infty$ as $k \to \infty$ such that $a_i^*(n+\delta_k) \to a_i^*(n), b_i^*(n+\delta_k) \to b_i^*(n), c_i^*(n+\delta_k) \to c_i^*(n), i=1,2$, as $k \to \infty$ for $n \in Z$. Suppose that $X = (x_1(n), x_2(n))$ is any solution of hull equation (5.1). By the proof of Lemma 2.2 and 2.3, we have

$$m_i \le \liminf_{n\to+\infty} x_i(n) \le \limsup_{n\to+\infty} x_i(n) \le M_i, \quad i=1,2. \tag{5.2}$$

And also

$$0 < \inf_{n\in Z^+} x_i(n) \le \sup_{n\in Z^+} x_i(n) < \infty, \qquad i=1,2.$$

Let ε be an arbitrary small positive number. There exists a positive integer n_0 such that $m_i - \varepsilon \le x_i(n) \le M_i + \varepsilon, n \ge n_0, i=1,2$. Write $X_k(n) = X(n+\tau_k) = (x_{1k}(n), x_{2k}(n))$, for all $n \ge n_0 + \tau - \tau_k, k \in Z^+$. We claim that there exists a sequence $\{y_i(n)\}$, and a subsequence of $\{\tau_k\}$, we still denote by $\{\tau_k\}$ such that $x_{ik}(n) \to y_i(n)$, uniformly in n on any finite subset B of Z as $k \to \infty$, where $B = \{a_1, a_2, \ldots, a_m\}, a_h \in Z (h=1,2,\ldots,m)$ and m is a finite number.

In fact, for any finite subset $B \subset Z$, when k is large enough, $\tau_k + a_h - \tau > n_0, h=1,2,\ldots,m$. So

$$m_i - \varepsilon \le x_i(n+\tau_k) \le M_i + \varepsilon, \quad i=1,2,$$

that is, $\{x_i(n+\tau_k)\}$ are uniformly bounded for large enough k.

Now, for $a_1 \in B$, we can choose a subsequence $\{\tau_k^{(1)}\}$ of $\{\tau_k\}$ such that $\{x_i(a_1+\tau_k^{(1)})\}$ uniformly converges on Z^+ for k large enough.

Similarly, for $a_2 \in B$, we can choose a subsequence $\{\tau_k^{(2)}\}$ of $\{\tau_k^{(1)}\}$ such that $\{x_i(a_2+\tau_k^{(2)})\}$ uniformly converges on Z^+ for k large enough.

Repeating this procedure, for $a_m \in B$, we can choose a subsequence $\{\tau_k^{(m)}\}$ of $\{\tau_k^{(m-1)}\}$ such that $\{x_i(a_m+\tau_k^{(m)})\}$ uniformly converges on Z^+ for k large enough.

Now pick the sequence $\{\tau_k^{(m)}\}$ which is a subsequence of $\{\tau_k\}$, we still denote it as $\{\tau_k\}$, then for all $n \in B$, we have $x_i(n+\tau_k) \to y_i(n)$ uniformly in $n \in B$, as $k \to \infty$.

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

By the arbitrary of B, the conclusion is valid.

Combined with

$$\begin{cases} x_{1k}(n+1) = x_{1k}(n)\exp\left\{a_1^*(n+\tau_k) - b_1^*(n+\tau_k)x_{1k}(n-\sigma_1) + c_1^*(n+\tau_k)\dfrac{x_{2k}(n-\tau_1)}{1+x_{2k}(n-\tau_1)}\right\}, \\ x_{2k}(n+1) = x_{2k}(n)\exp\left\{a_2^*(n+\tau_k) - b_2^*(n+\tau_k)x_{2k}(n-\sigma_2) + c_2^*(n+\tau_k)\dfrac{x_{1k}(n-\tau_2)}{1+x_{1k}(n-\tau_2)}\right\}, \end{cases}$$

gives

$$\begin{cases} y_1(n+1) = y_1(n)\exp\left\{a_1^*(n) - b_1^*(n)y_1(n-\sigma_1) + c_1^*(n)\dfrac{y_2(n-\tau_1)}{1+y_2(n-\tau_1)}\right\}, \\ y_2(n+1) = y_2(n)\exp\left\{a_2^*(n) - b_2^*(n)y_2(n-\sigma_2) + c_2^*(n)\dfrac{y_1(n-\tau_2)}{1+y_1(n-\tau_2)}\right\}, \end{cases}$$

We can easily see that $Y(n) = (y_1(n), y_2(n))$ is a solution of hull equation (5.1) and $m_i - \varepsilon \le y_i(n) \le M_i + \varepsilon, i = 1, 2$, for $n \in Z$. Since ε is an arbitrary small positive number, it follows that $m_i \le y_i(n) \le M_i, i = 1, 2$, for $n \in Z$, that is

$$0 < \inf_{n\in Z} y_i(n) \le \sup_{n\in Z} y_i(n) < \infty, \quad i = 1, 2.$$

Hence each hull equation of almost periodic difference system (1.2) has at least one strictly positive solution.

Now we prove the uniqueness of the strictly positive solution of each hull equation (5.1). Suppose that the hull equation (5.1) has two arbitrary strictly positive solutions $(x_1^*(n), x_2^*(n))$ and $(y_1^*(n), y_2^*(n))$. Like in the proof of Theorem 4.1, we construct a Lyapunov functional

$$V^*(n) = \sum_{i=1}^{2} \beta_i\Big(V_{i1}^*(n) + V_{i2}^*(n) + V_{i3}^*(n)\Big), \quad n \in Z, \tag{5.3}$$

where

$$\begin{aligned} V_{i1}^*(n) &= |\ln x_i^*(n) - \ln y_i^*(n)| \\ V_{i2}^*(n) &= \sum_{s=n-\tau_i}^{n-1} c_i(s+\tau_i)\big|x_j^*(s) - y_j^*(s)\big| \\ &\quad + \sum_{s=n}^{n-1+\sigma_i} b_i(s) \sum_{u=s-\sigma_i}^{n-1} \Big\{A_i(u)[a_i(u) + M_i b_i(u) + M_j c_i(u)]\big|x_i^*(u) - y_i^*(u)\big| \\ &\quad + M_i B_i(u)c_i(u)\big|x_j^*(u-\tau_i) - y_j^*(u-\tau_i)\big| \\ &\quad + M_i B_i(u)b_i(u)\big|x_i^*(u-\sigma_i) - y_i^*(u-\sigma_i)\big|\Big\}, \\ V_{i3}^*(n) &= M_i \sum_{l=n-\tau_i}^{n-1} B_i(l+\tau_i)c_i(l+\tau_i)\big|x_j^*(l) - y_j^*(l)\big| \sum_{s=l+\tau_i+1}^{l+\tau_i+\sigma_i} b_i(s) \\ &\quad + M_i \sum_{l=n-\sigma_i}^{n-1} B_i(l+\sigma_i)b_i(l+\sigma_i)\big|x_i^*(l) - y_i^*(l)\big| \sum_{s=l+\sigma_i+1}^{l+2\sigma_i} b_i(s), \quad i, j = 1, 2, i \ne j. \end{aligned}$$

Calculating the difference of V^* along the solution of the hull equation(5.1), like in the discussion of (4.13), one has

$$\Delta V^* \le -\eta \sum_{i=1}^{2} |x_i^*(n) - y_i^*(n)|, \quad n \in Z. \tag{5.4}$$

From (5.4), we can see that $V^*(n)$ is a non-increasing function on Z. Summing both sides of the above inequalities from n to 0, we have

$$\eta \sum_{k=n}^{0} \sum_{i=1}^{2} |x_i^*(k) - y_i^*(k)| \le V^*(0) - V^*(n+1), \quad n < 0.$$

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

Note that $V^*(n)$ is bounded. Hence we have

$$\sum_{k=-\infty}^{0}\sum_{i=1}^{2}|x_i^*(k)-y_i^*(k)|<+\infty,$$

which implies that

$$\lim_{n\to\infty}|x_i^*(n)-y_i^*(n)|=0,\quad i=1,2. \tag{5.5}$$

Define $Q=\sum_{i=1}^{2}\alpha_i Q_i$, where

$$\begin{aligned}Q_i=&\frac{1}{m_i}+\tau_i c_i^u+\sigma_i^2 b_i^u[A_i^u(a_i^u+M_i b_i^u+M_j c_i^u)+M_i B_i^u(c_i^u+b_i^u)]\\&+\sigma_i M_i B_i^u(\tau_i c_i^u+\sigma_i b_i^u),\quad i,j=1,2,i\neq j.\end{aligned}$$

Let ε be an arbitrary small positive number. It follows from (5.5) that there exists a positive integer $n_1>0$ such that $|x_i^*(n)-y_i^*(n)|<\dfrac{\varepsilon}{Q},n<-n_1,i=1,2$. Therefore, for $n<-n_1,i,j=1,2,i\neq j$,

$$\begin{aligned}V_{i1}^*(n)\leq&\frac{1}{m_i}|x_i^*(n)-y_i^*(n)|\leq\frac{1}{m_i}\frac{\varepsilon}{Q},\\V_{i2}^*(n)\leq&\tau_i c_i^u\frac{\varepsilon}{Q}+\sigma_i^2 b_i^u\Big[A_i^u(a_i^u+M_i b_i^u+M_j c_i^u)\max_{p\leq n}|x_i^*(p)-y_i^*(p)|\\&+M_i B_i^u c_i^u\max_{p\leq n}|x_j^*(p)-y_j^*(p)|+M_i B_i^u b_i^u\max_{p\leq n}|x_i^*(p)-y_i^*(p)|\Big]\\&\leq\Big\{\tau_i c_i^u+\sigma_i^2 b_i^u[A_i^u(a_i^u+M_i b_i^u+M_j c_i^u)+M_i B_i^u(c_i^u+b_i^u)]\Big\}\frac{\varepsilon}{Q},\\V_{i3}^*(n)\leq&\sigma_i\tau_i M_i c_i^u B_i^u\max_{p\leq n}|x_j^*(p)-y_j^*(p)|+\sigma_i^2 M_i b_i^u B_i^u\max_{p\leq n}|x_i^*(p)-y_i^*(p)|\\&\leq\sigma_i M_i B_i^u(\tau_i c_i^u+\sigma_i b_i^u)\frac{\varepsilon}{Q}.\end{aligned}$$

It follows from (5.3) and above inequalities that

$$V^*(n)\leq\sum_{i=1}^{2}\beta_i Q_i\frac{\varepsilon}{Q}=\varepsilon,\ n<-n_1,$$

so $\lim\limits_{n\to-\infty}V^*(n)=0$. Note that $V^*(n)$ is a non-increasing function on Z, and then $V^*(n)\equiv 0$. that is $x_i^*(n)=y_i^*(n),i=1,2$, for all $n\in Z$, Therefore, each hull equation of system (1.2) has a unique strictly positive solution.

In view of the above discussion, any hull equation of system (1.2) has a unique strictly positive solution. By Lemma 2.2-2.3 and Theorems 4.1, the almost periodic difference system (1.2) has a unique strictly positive almost periodic solution which is globally attractive. The proof is completed. □

Let $\tau_{ij}=0,i=1,2,j=0,1,2$. Like in the proof of Theorem 4.1, we have the following corollary.

Corollary 5.1 Let $\sigma_1=\sigma_2=\tau_1=\tau_2=0$. Assume that there exist positive constants β_1 and β_2, such that

$$\beta_i\min\{b_i^l,\frac{2}{M_i}-b_i^u\}-\beta_j c_j^u>0,\quad i,j=1,2,\ i\neq j,$$

where $M_i=\dfrac{1}{b_i^l}\exp\left\{a_i^u+b_i^u-1\right\}$. Then the almost periodic difference system (1.1) admits a unique strictly positive almost periodic solution, which is globally attractive.

World Open Journal of Advanced Mathematics
Vol. 3, No. 4, December 2016, pp. 1-14
Available online at http://scitecpub.com/Journals.php

6 Example and numerical simulation

In this section, we give the following example to check the feasibility of our result.

Example Consider the following almost periodic discrete Lotka-Volterra mutualism model with delays:

$$\begin{cases} x_1(n+1) = x_1(n)\exp\Big\{0.025 + 0.005\sin(n) - (1.0075 - 0.0025\cos(n))x_1(n-1) \\ \qquad\qquad + (0.03 - 0.005\cos(n))\dfrac{x_2(n-2)}{1+x_2(n-2)}\Big\}, \\ x_2(n+1) = x_2(n)\exp\Big\{0.015 + 0.005\cos(n) - (1.15 + 0.05\sin(n))x_2(n-1) \\ \qquad\qquad + (0.02 - 0.005\sin(n))\dfrac{x_1(n-2)}{1+x_1(n-2)}\Big\}. \end{cases} \tag{6.1}$$

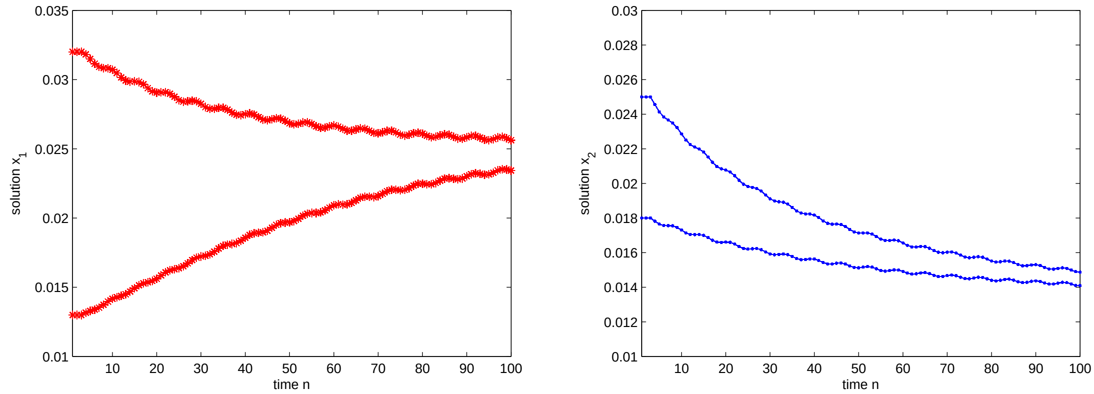

FIGURE1: Dynamic behavior of system (6.1) with the initial conditions $(x_1(n), x_2(n)) = (0.013, 0.018)$ and $(0.032, 0.025), n = 1, 2, 3$ for $k \in [1, 100]$, respectively.

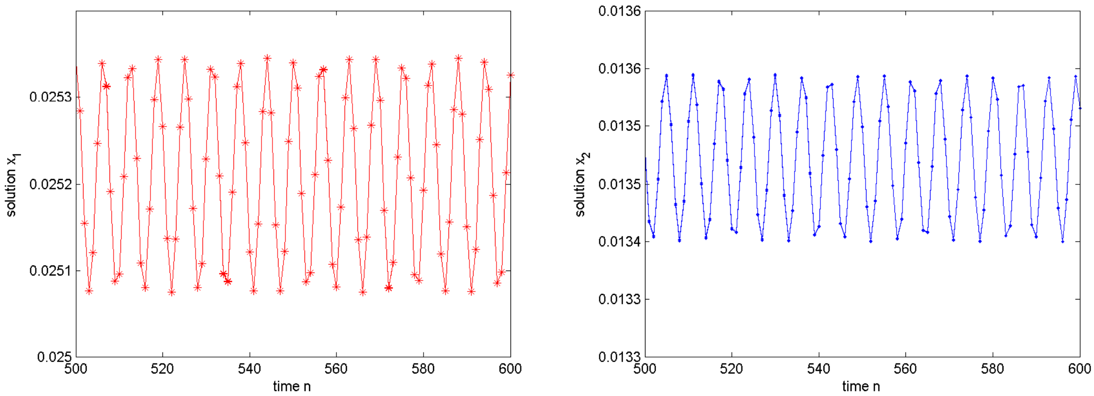

FIGURE2: Dynamic behavior of system (6.1) with the initial conditions $(x_1(n), x_2(n)) = (0.013, 0.018)$ and $(0.032, 0.025), n = 1, 2, 3$ for $k \in [500, 600]$, respectively.

By simple computation, we derive

$$M_1 \approx 0.4169, \quad M_2 \approx 0.3659, \quad E_{12} \approx 0.4893, \quad E_{21} \approx 0.4992, \quad F_1 \approx 0.0507, \quad F_2 \approx 0.0365.$$

Then

$$E_{12} - F_2 \approx 0.4528 > 0.4, \quad E_{21} - F_1 \approx 0.4485 > 0.4.$$

Also it is easy to see that the condition (4.6) is verified. Therefore, system (6.1) has a unique strictly positive almost periodic solution which is globally attractive. Our numerical simulations support our results(see Figs.1 and 2).

7 Acknowledgements

This work is supported by Scientific Research Program Funded by Shaanxi Provincial Education Department of China (No.16JK1696). There are no financial interest conflicts between the authors and the commercial identity.

References

[1] Y. Li, Positive periodic solutions of a discrete mutualism model with time delays, Int. J. Math. Math. Sci. 4 (2005)499-506.

[2] F. Chen, Permanence for the discrete mutualism model with time delays, Math. and Comp. Modelling. 47(2008) 431-435.

[3] Y. Li and G. Xu, Positive periodic solutions for an integrodifferential model of mutualism, Appl. Math. Lett. 14 (5)(2001)525-530.

[4] X. Yan and W. Li, Bifurcation and global periodic solutions in a delayed facultative mutualism system, Physica D. 227(2007)51-69.

[5] Y. Li and H. Zhang, Existence of periodic solutions for a periodic mutualism model on time scales, J. Math. Anal. Appl. 343(2008)818-825.

[6] X. Chen and F. Chen, Almost periodic solutions of a delay population equation with feedback control, Nonlinear Anal. RWA B7(2006)559-571.

[7] X. Chen, Almost periodic solutions of a nonlinear delay population equation with feedback control, Nonlinear Anal. RWA B8(2007)62-72.

[8] C.H. Feng, On the existence and uniqueness of almost periodic solutions for delay Logistic equations, Appl. Math. Comput, 136(2-3)(2003)487-494.

[9] K. Gopalsamy, Global asymptotic stability in an almost periodic Lotka-Volterra system, J. Austral. Math. Soc. Ser. B27(1986)346-360.

[10] Z. Teng, Nonautonomous Lotka-Volterra systems with delays, J. Differential Equations, 179(2002)538-561.

[11] Z. Teng, L. Chen, Permanence and extinction of periodic predator-prey systems in a patchy environment with delay, Nonlinear, Anal, 4(2003)335-364.

[12] Z. Teng, On the non-autonomous Lotka-Volterra N-species competing systems, Appl. Math. Comput. 114(2000)175-185.

[13] Z. Li and F. Chen, Almost periodic solutions of a discrete almost periodic logistic equation, Math. Comput. Modelling, 50(2009)254-259.

[14] C. Niu and X. Chen, Almost periodic sequence solutions of a discrete Lotka–Volterra competitive system with feedback control, Nonlinear Anal. RWA 10(2009)3152-3161.

[15] T. Zhang, Y. Li and Y. Ye, Persistence and almost periodic solutions for a discrete fishing model with feedback control, Commun. Nonlinear Sci. Numer. Simul. 16(2011)1564-1573.

[16] Z. Li, F. Chen and M. He, Almost periodic solutions of a discrete Lotka-Volterra competition system with delays, Nonlinear Anal. RWA, 12(2011)2344-2355.

[17] L. Wu, F. Chen and Z. Li, Permanence and global attractivity of a discrete Schoener's competition model with delays, Math. and Comp. Modelling. 49(2009)1607-1617.

[18] S. Zhang and G. Zheng, Almost periodic solutions of delay difference systems, Appl. Math. Comput. 131(2002)497-516.

[19] A.M. Fink, G. Seifert, Liapunov functions and almost periodic solutions for almost periodic systems, J. Differential Equations, 5(1969)307-313.

[20] Y. Hamaya, Existence of an almost periodic solution in a difference equation by Liapunov functions, Nonlinear Stud. 8(3)(2001)373-379.

[21] S.N. Zhang, Existence of almost periodic solution for difference systems, Ann. Differential Equations 16(2)(2000)184-206.

[22] R. Yuan and J. Hong, The existence of almost periodic solutions for a class of differential equations with piecewise constant argument, Nonlinear Anal. 28(8)(1997)1439-2450.